

Exactly solvable pairing models inspired by high- T_c superconductivity

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The BCS theory models electron correlations with pure zero-momentum pairs. I consider a family of pairing Hamiltonians, where the electron correlations are modelled with pure arbitrary-momentum pairs. I find unexpectedly that all models in this family are exactly solvable. I present these exact solutions. The BCS pairing, the η pairing and the d -wave pairing in T_c superconductivity are the family members. These models, expect for the BCS model, are legitimate only on an xy plane defined by the electron spin S_z , compatible with the two dimensionality of high temperature superconductivity. Surprisingly, all pairings are on equal footing in the xy plane, suggesting a unification of the s -wave and d -wave theoretical mechanisms in high T_c superconductivity. I also give the extension that includes all members of this family.

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Introduction.— One of the major unsolved problems of the theoretical condensed matter physics is how superconductivity arises in high-temperature superconductors. The mechanism of electron pair formation in these crystals is still obscure. Although there have been many promising leads, a real explanation remains exclusive or even controversial. One of the prime reasons for this is that the superconductive materials are normally very complex, multi-layered crystals such that it is hard to have accurate theoretical solutions either from exactly solvable models or trustworthy numerical methods on such strongly interacting electron systems.

There have been two representative theories for the unconventional superconductivity, the d -wave pairing related to antiferromagnetic spin fluctuations [1] and the modified BCS-type (s -wave) pairing [2]. None of them are fully accepted, but both of them are associated with sort of pairing mechanism.

In this letter, I start with a general electron Hamiltonian and consider a family of pairing models, including those with the BCS pairs [3], the η pairs [4] and the d -wave pairs [5, 6] in T_c superconductivity. I find exact solutions to this family, with the assistance of the dressing transformations introduced for qubits [7]. The total electron spin \vec{S} of these models are not conserved except for the BCS model, while the conservation of the z -component S_z of the total spin remains. These models are legitimate as physical Hamiltonians only on an xy - plane defined by S_z , which is compatible with the two dimensionality of high T_c superconductivity. Different pairings in these models are on equal footing in the sense that they can be transformed unitarily into each other. In particular, these models can be converted to the conventional BCS model, with an interesting extra single-particle energy. I study this extra contribution of single-particle energy and observe that it plays a decisive role in a transition between independent-particle ground states and collective η -pairing or d -wave pairing ground states. I find conditions where non-BCS pairs, including the η -pairs and the d -wave pairs, play an identical role as the BCS pairs. In this case, the ansatz in constructing the BCS model implies that these non-BCS pairs

could become dominant.

General electron Hamiltonian, pairs with given momentum and $SU(2)$ symmetry – Consider a general Hamiltonian of electrons on a periodic lattice.

$$H = \sum_{k, \sigma=\downarrow, \uparrow} \epsilon_k n_{k\sigma} + V, \quad (1)$$

where $n_{k\sigma} = c_{k\sigma}^\dagger c_{k\sigma}$ is the number operator of an electron with momentum vector k and spin σ . The electron-electron interaction is

$$V = \sum_{q, k, k'} V_{k, k'} c_{q-k'}^\dagger c_{k'}^\dagger c_{k\uparrow} c_{q-k\downarrow}. \quad (2)$$

Here $c_{k\uparrow}(c_{k\downarrow})$ is a momentum-space annihilation operator of a spin-up (spin-down) electron. The lattice can be one-, two- or three- dimensional with the total number L of lattice sites in each dimension. The vectors or modes $k = (k_x, k_y, k_z)$ with $k_{x,y,z} = 2\pi l/L$, where $l = 0, 1, \dots, L-1$. Vectors q have the same modes as k . The interaction can be rewritten as

$$V = \sum_{q, k, k'} V_{k, k'} \eta_q^\dagger(k') \eta_q(k), \quad (3)$$

which is expressed in terms of pair operators

$$\eta_q(k) = c_{k\uparrow} c_{q-k\downarrow} \text{ and } \eta_q^\dagger(k) = c_{q-k\downarrow}^\dagger c_{k\uparrow}^\dagger. \quad (4)$$

It is easy to check that, for a given q , these operators satisfy commutation relations,

$$[\eta_q(k), \eta_q^\dagger(k')] = \delta_{kk'} (1 - c_{k\uparrow}^\dagger c_{k\uparrow} - c_{q-k\downarrow}^\dagger c_{q-k\downarrow}), \quad (5)$$

such that $\eta_q(k)$, $\eta_q^\dagger(k)$ and $[\eta_q(k), \eta_q^\dagger(k)]$ close an $su(2)$ algebra. In other words, they play the same roles as Pauli matrices[8],

$$\begin{aligned} \sigma_k^- &\Longleftrightarrow \eta_q(k), \\ \sigma_k^+ &\Longleftrightarrow \eta_q^\dagger(k), \\ \sigma_k^z &\Longleftrightarrow (c_{k\uparrow}^\dagger c_{k\uparrow} + c_{q-k\downarrow}^\dagger c_{q-k\downarrow} - 1)/2, \end{aligned} \quad (6)$$

acting on the two bases $|0\rangle_k = I_k |0\rangle$ and $|1\rangle_k = \eta_q^\dagger(k) |0\rangle$, where $|0\rangle$ is the vacuum state and I_k is a unit operator on the k -th mode. These define a qubit, or precisely the k -th qubit in a *collective* subspace of the entire electron Hilbert space.

In case that $V_{k-k'} = G$ is constant, the interaction $V = G \sum_{q,k,k'} \eta_q^\dagger(k') \eta_q(k)$ corresponds to a one-band Hubbard-like model [9], which is a sum of different q -components $\sum_{k'} \eta_q^\dagger(k') \sum_k \eta_q(k)$ (Note that here q is not summed over). In contrast, the BCS ansatz considers particularly the $q = 0$ component and also sets $V_{k,k'} = G$, such that $V = G \sum \eta_0^\dagger(k') \eta_0(k)$. Here G is negative for the BCS model.

Simplified interactions: pure pairing models.— Generally, pairs $\eta_q(k)$ and $\eta_{q'}^\dagger(k)$ for different values of q do not commute but are related. As an ansatz, the BCS theory chooses the simplest $q = 0$ pair component and neglects the others, which has been verified by numerous experiments. Motivated by this BCS (pure-pair) ansatz, I consider the family of all pure q -components in the general Hamiltonian (1). I also employ a slightly general separable couplings $V_{k,k'} = Gg(k)g(k')$, where the BCS assumption $V_{k,k'} = G$ is a special case when $g(k) = g(k') = 1$. The family of pure q -component models can therefore be written as

$$H_q = \sum_{k,\sigma=\downarrow,\uparrow} \epsilon_k n_{k\sigma} + V_q, \quad (7)$$

where $V_q = G\eta_q^\dagger \eta_q$ and q run over the total momentum space. This letter will mostly concentrate on the cases with $G < 0$ as in the BCS theory. I term operators η_q as q -pairs, which are defined as

$$\eta_q = \sum_k g(k) c_{k\uparrow} c_{q-k\downarrow}, \quad (8)$$

$$\eta_q^\dagger = \sum_k g(k) c_{q-k\downarrow}^\dagger c_{k\uparrow}^\dagger, \quad (9)$$

behaving as *collective* pairs. I also call the pure q -component models H_q as pure q -pairing models. The pair with $q = 0$ and $g(k) = 1$ corresponds to the BCS collective pair.

It is interesting to note that the η_π pair with $g(k) = 1$ is the η pair introduced in [4], but it corresponds to the d -wave pair [5, 6] when

$$g(k) = \text{sign}(\cos k_x - \cos k_y) = \pm 1, \quad (10)$$

recommended in ref. [10]. In order for $\pi - k$ and k to be simultaneously possible k values, the number of sites, L , must be *even* for the η pair [4]. I apply this constraint to arbitrary value of $q = 2\pi/L$ (three-dimensional integer) $(\text{mod } 2\pi)$, requiring that the even-odd parity of L is the same as that of $2\pi/q$, otherwise $q - k$ and k would not be simultaneously possible k values.

Symmetry constrains.— The total spin operator of electrons is

$$\vec{S} = \frac{1}{2} \sum_{k\alpha\beta} c_{k\alpha}^\dagger \vec{\sigma}_{\alpha\beta} c_{k\beta},$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli's matrices. The BCS model H_0 conserves the total spin \vec{S} , i.e., $[H_0, \vec{S}] = 0$. However, while S_z remains conserved, the total spin is no longer conserved for the rest of $q \neq 0$ pairing models (7). It indicates that $H_{q \neq 0}$ works in an anisotropic spin space, or a two-dimensional xy -plane perpendicular to the z -axis in spin space. Quantum states are common eigenvectors of S_z and $H_{q \neq 0}$. I have not found further coordinate space symmetry for $H_{q \neq 0}$, however an assumption of electrons being on the xy plane of the coordinate space is clearly consistent with the conservation of S_z .

Dressing transformation and the BCS model.— I now come to the main results of this letter. Specifically, I first introduce the following dressing transformations [7]

$$\mathcal{W}_{qq'} = \exp\left(-\frac{\pi}{2} \left[\sum_k g(k) (c_{q-k\downarrow}^\dagger c_{q'-k\downarrow} - c_{q'-k\downarrow}^\dagger c_{q-k\downarrow}) \right] \right), \quad (11)$$

where I set $g^2(k) = 1$ as done in [10] and will allow it to be an arbitrary real function later. One can check that any two q -pairs, η_q pairs and $\eta_{q'}$ pairs, can be transformed or rotated unitarily into each other, i.e., $\eta_{q'} = \mathcal{W}_{qq'}^\dagger \eta_q \mathcal{W}_{qq'}$. These unitary dressing transformations do not change the $su(2)$ commutation relation (5) but rotate the bases, $\eta_q^\dagger(k) |0\rangle \rightarrow \eta_{q'}^\dagger(k) |0\rangle$. The forms of electron-electron interactions in H_q are invariant under these transformations,

$$V_{q'} = \mathcal{W}_{qq'}^\dagger V_q \mathcal{W}_{qq'}.$$

It also shows that pair correlations with different q are not independent. On the contrary, these correlations are equivalent, or *similar* in mathematical term, to the BCS pair correlation V_0 , subject to the unitary transformations \mathcal{W}_{q0} (notation as \mathcal{W}_q for simplicity). Both pairs must live on the two-dimensional spin space due to the conservation of S_z .

It is notable that under the dressing transformations (11), the forms of single-particle energies are likewise invariant,

$$\mathcal{W}_q^\dagger \sum_{k,\sigma=\downarrow,\uparrow} \epsilon_k c_{k\sigma}^\dagger c_{k\sigma} \mathcal{W}_q = \sum_{k,\sigma=\downarrow,\uparrow} \epsilon_{k\sigma}(q) c_{k\sigma}^\dagger c_{k\sigma},$$

where $\epsilon_{k\sigma}(q) = \epsilon_k \delta_{\sigma\uparrow} + \epsilon_{k-q} \delta_{\sigma\downarrow}$. It remains diagonal but becomes spin-dependent. This concludes that the q -pair Hamiltonian H_q is equivalent to the BCS Hamiltonian with a spin-dependent single-particle energy. The spin dependence can be separated from the total single-particle energy. Consequently, the q -pairing models are converted to an exactly same form as the BCS model with spin-independent levels $\epsilon_k(q)$ plus a single-particle Hamiltonian commutable with the electron-electron interaction. The effective levels now are functions of q . A q -pair model can therefore be treated as a BCS model H_0 with single-particle levels $\epsilon_k(q)$ plus an extra \hbar that commutes with H_0 .

Exact solutions for the family of q -pair Hamiltonians.— The equivalence or similarity leads to exact solutions of all models

in this family. The first step is to solve the eigenproblem of the BCS Hamiltonian $\tilde{H}_0 = \sum_k \epsilon_k(q)(n_{k\uparrow} + n_{k\downarrow}) + G\eta_0^\dagger \eta_0$,

$$\tilde{H}_0 |\Psi\rangle = E |\Psi\rangle,$$

where $\tilde{H}_0 + \bar{h} = \mathcal{W}_q^\dagger H_q \mathcal{W}_q$ and $\epsilon_k(q) = (\epsilon_k + \epsilon_{k-q})/2$ are effective single-particle levels. The extra single-particle energy $\bar{h} = \sum_k (\epsilon_k - \epsilon_{k-q})(n_{k\uparrow} - n_{k\downarrow})$ commutes with \tilde{H}_0 , such that common eigenfunctions of \tilde{H}_0 and \bar{h} are allowed. Note that I have rewritten the single-particle energy of H_q as

$$\begin{aligned} & \sum_{k,\sigma=\downarrow,\uparrow} \epsilon_{k\sigma}(q) c_{k\sigma}^\dagger c_{k\sigma} \\ &= \bar{h} + \sum_k \epsilon_k(q) (c_{k\uparrow}^\dagger c_{k\uparrow} + c_{k\downarrow}^\dagger c_{k\downarrow}). \end{aligned}$$

The q -pair expression of \bar{h} is $\bar{h}_q = \mathcal{W}_q \bar{h} \mathcal{W}_q^\dagger$.

Having common eigenstates $|\Psi\rangle$ of \tilde{H}_0 and \bar{h} , one can obtain the eigenstates $|\Psi_q\rangle$ by the inverse transformation of \mathcal{W}_q ,

$$|\Psi_q\rangle = \mathcal{W}_q |\Psi\rangle,$$

specifically, replacing all $c_{k\downarrow}$ and $c_{k\downarrow}^\dagger$ with $c_{q-k\downarrow}$ and $c_{q-k\downarrow}^\dagger$ in $|\Psi\rangle$ to obtain $|\Psi_q\rangle$. The corresponding eigenenergies E remain unchanged.

The exact solution of H_q with N electrons can be written as [11]

$$|\Psi_q\rangle = \prod_{l=1}^M S_l^\dagger |m\rangle, S_l^\dagger = \sum_k \frac{1}{2\epsilon_k(q) - E_l} g(k) \eta_q^\dagger(k), \quad (12)$$

where $2M = N - m$ and $m = \sum m_k$ is the number of unpaired electrons, defined by $\eta_q(k) |m\rangle = 0$ and $(c_{k\uparrow}^\dagger c_{k\uparrow} + c_{q-k\downarrow}^\dagger c_{q-k\downarrow}) |m\rangle = m_k |m\rangle$, in particular $m = 0$ for the ground state. It may deserve mentioning that the $m = 0$ subspace is in one to one correspondence with the entire Hilbert space of qubits. It suggests that a superconductor may act as a natural quantum computer.

E_l satisfy the Richardson's equation,

$$2 \sum_{l \neq m} \frac{1}{E_m - E_l} - \sum_k \frac{1 - m_k}{2\epsilon_k(q) - E_l} = \frac{1}{G},$$

and the eigenenergies of H_q are

$$E = \bar{E}_q + \sum_k \epsilon_k(q) m_k + \sum_{l=1}^M E_l, \quad (13)$$

where \bar{E}_q are eigenvalues of \bar{h} and are given by simply filling the spin-dependent single-particle levels $(\epsilon_k - \epsilon_{k-q})$ for spin-up and $-(\epsilon_k - \epsilon_{k-q})$ for spin-down.

The standard BCS treatment may also be interesting in q -pairing models since it is a good approximation for large systems. The BCS approximate solution is

$$E_{BCS} \approx \bar{E}_q + 2 \sum_k (\epsilon_k(q) - \lambda) v_k^2 + \Delta^2/G,$$

where the terms with v^4 are neglected as usual. The BCS wave function is

$$|\Psi_q\rangle = \prod (u_k + v_k g(k) \eta_q^\dagger(k)) |0\rangle,$$

which has $d_{x^2-y^2}$ pairing symmetry when $g(k)$ in eq. (10) is taken [10]. The gap parameter $\Delta = |G| \sum v_k u_k$ and

$$\left. \begin{matrix} u_k^2 \\ v_k^2 \end{matrix} \right\} = \frac{1}{2} \pm \frac{\epsilon_k(q) - \lambda}{\sqrt{(\epsilon_k(q) - \lambda)^2 + \Delta^2}}.$$

Again, there are single particle energies \bar{E}_q , and for the BCS pairing the energy $\bar{E}_0 = 0$.

The expectation values of an observable are subject to the same dressing transformations,

$$\langle \Psi_q | O_q | \Psi_q \rangle = \langle \Psi | \mathcal{W}_q^\dagger O_q \mathcal{W}_q | \Psi \rangle, \quad (14)$$

which are well-defined and can be obtained via the known BCS theoretical methods. There are observables invariant under the dressing transformations, for instance, the z -component of the total spin $\mathcal{W}_q^\dagger S_z \mathcal{W}_q = S_z$. The charge density wave operator $Q_+ = \sum_k (c_{q-k\uparrow}^\dagger c_{k\uparrow} + c_{q-k\downarrow}^\dagger c_{k\downarrow})$ [5] has this property as well, $\mathcal{W}_q^\dagger Q_+ \mathcal{W}_q = Q_+$.

Illustration— I would like emphasize that the η pairs or d -wave pairs can be treated on the equal footing in this framework, as long as the different $g(k)$ are used. It is interesting to note for the η pairs or d -wave pairs that $\epsilon_k(\pi) = 4\epsilon$ if one uses the single particle levels $\epsilon_k = 4\epsilon - 2\epsilon \cos k_x - 2\epsilon \cos k_y$ in the two-dimensional attractive Hubbard model. It will contribute a trivial energy $4\epsilon N$ and will be neglected in the follow discussions. The solution of H_π is straightforward since $[H_\pi, G\eta_\pi^\dagger \eta_\pi] = 0$.

Without loss of generality, we consider the one-dimensional case. The ground state is a result of the competition between the pairing energy

$$E_p(N) = -UN(2L - N + 2)/L,$$

of $G\eta_\pi^\dagger \eta_\pi$ ($G = -U/L$ is used) and the single-particle energies $E_\pi(N)$ given by $\bar{h} = -2\epsilon \sum_k \cos k (n_{k\uparrow} - n_{k\downarrow})$. The energy before half-filling is

$$\bar{E}_\pi(K) = -4\epsilon \frac{\sin(K+1)a + \sin(Ka) + \sin a}{\sin a},$$

where $a = 2\pi/L$ and $N = 4K$ since \bar{h} has four-fold degeneracy. The total energy at half filling is $E_s(L) \approx -2\epsilon L/\pi$. After half-fill it becomes

$$\bar{E}_\pi(K) = 4\epsilon \frac{\sin((K+1)a) - \sin(aK) - \sin a}{\cos a - 1},$$

where $(N - L) = 4K$. Numerical calculation shows that when U/ϵ is small, \bar{h} is dominate in the ground state competition, where spins arrange themselves up and down such that the total magnetization is always zero for $N = 4K$. I will

call this state as MZ independent-particle state or MZ state. For big values of U/ϵ , the π pair (or d -wave pair) correlation $G\eta_\pi^\dagger\eta_\pi$ becomes dominate. The ground state therefore is in η_π^\dagger "condensation",

$$|\Psi_\pi\rangle = \frac{1}{\sqrt{N!(L-N)!}} \eta_\pi^{\dagger N} |v\rangle.$$

When the values of U/ϵ are inbetween, the ground state is in the MZ state around half-filling but becomes η_π^\dagger condensation as the particle or hole number gets smaller. For instance, when $L = 10^4$, $U/\epsilon < 1$ corresponds to the MZ state and $U/\epsilon > 1.31$ to the η_π^\dagger condensation. When $1 > U/\epsilon > 1.31$, the ground states is the MZ states around half-filling and becomes η_π^\dagger condensation when particle or hole number is small. Specifically, $U/\epsilon = 1.26$, the ground state leaves the MZ state and becomes η_π^\dagger condensation at the doping level 0.2 and until full filling.

The single particle energy \bar{h} plays the essential role in this analysis and the conclusions here are applicable to both the η pairs and d -wave pair, depending only on $g(k)$. (I noticed after finishing the second version that solutions with $g(k) = 1$ are considered in previous works [12] for the FFLO states).

Extensions.— It is instructive to rewrite the Hubbard-type interaction as

$$V = G \sum_{q=0} \mathcal{W}_q \eta_0^\dagger \eta_0 \mathcal{W}_q^\dagger, \quad (15)$$

where $\mathcal{W}_0^\dagger = 1$. The expectation values of V under a wave function $|\Phi\rangle$ is $\langle\Phi|V|\Phi\rangle = G\text{Tr}(\eta_0^\dagger\eta_0\rho)$, where

$$\rho = \sum_{q=0} \mathcal{W}_q^\dagger |\Phi\rangle \langle\Phi| \mathcal{W}_q, \quad (16)$$

where ρ is a non-normalized density matrix. This is similar to the Kraus representation with many *channels*.

The crucial ansatz in the BCS theory is to pick up one pure channel, the BCS pair *channel* at $q = 0$. The ansatz has been verified by numerous experiments in normal superconductivity. This indicates clearly that the BCS pairs η_0^\dagger are dominate in low-lying states in normal superconductors. Since all pairs in the interaction are on equal footing, intuitively the single particle energy should be responsible for validity of the BCS ansatz. In other words, the single particle levels ϵ_k are in favour of the BCS ansatz. Note that the single particle energy $h = \sum \epsilon_k c_{k\sigma}^\dagger c_{k\sigma}$, in the above discussions, commutes with the total spin \vec{S} .

Heretofore, I have not made any physical assumption except generalizing and studying the BCS ansatz to q -pairs. There are evidences that all known high-temperature superconductors are strongly two-dimensional. Since the $q \neq 0$ pairing Hamiltonians keep the conservation of S_z and are not contradict with these evidences, one may extend the forms of the single-particle levels to spin-dependent ones, for instance considering the effect from spin-orbital coupling, and writes

them as $\sum_{k,\sigma=\downarrow,\uparrow} e_{k\sigma} c_{k\sigma}^\dagger c_{k\sigma}$. It and the total q -pair Hamiltonian H_q commute with S_z . The dressing transformation \mathcal{W}_q will therefore give

$$\begin{aligned} \eta_q &\rightarrow \eta_0, \\ e_{k\uparrow} &\rightarrow e_{k\uparrow}, \\ e_{k\downarrow} &\rightarrow e_{k-q\downarrow}. \end{aligned} \quad (17)$$

In the case that $e_{k\downarrow} = e_{k-q\downarrow} = \epsilon_k$, η_q will play the exactly same role as η_0 . This requires, for the attractive Hubbard model, $e_{k\uparrow} = 4\epsilon - 2\epsilon \cos k_x - 2\epsilon \cos k_y$ and $e_{k\downarrow} = -e_{k\uparrow}$, when $q = \pi$. The η pairs or d -wave pairs are therefore in the same position as the BCS pairs η_0 , while η_0 pairs behave the same as η_π pairs as discussed in the last section. In this case, the BCS ansatz implies that the η pairs or d -wave pairs are dominate in low-lying states.

Another possible extension is to release the constraints of functions $g(k)$ in the dressing transformations (11). Models generated by any form of $g(k)$ are still in the exactly solvable family since they can be unitarily rotated to the BCS model. For instance when $g(k) = 2\theta/\pi$ and $q = \pi$, a hybrid model

$$\begin{aligned} H &= \cos^2 \theta H_0 + \sin^2 \theta H_\pi \\ &+ \frac{\cos 2\theta}{2} \left(\sum_k \epsilon_k c_{q+k\downarrow}^\dagger c_{k\downarrow} + G\eta_0^\dagger \eta_\pi + h.c. \right), \end{aligned} \quad (18)$$

is equivalent to the BCS model and may be used to explain the competition between the BCS pairs and η -pairs. Note that now there is a scattering term between the BCS pairs and η -pairs.

The most general extensions.— Generally, all Hamiltonians $H = W(H_0 + h_B)W^\dagger$ with an arbitrary dressing transformation W are in the exactly solvable family, where h_B commutes with the BCS Hamiltonian H_0 and may even be for another system such as a bath. The eigenwavefunctions of H are $W|\Psi\rangle|B\rangle$, where $|\Psi\rangle$ are the BCS eigenfunctions and if there is a bath, $|B\rangle$ are eigenfunctions of h_B . Eigenvalues are $E + E_B$, which are sums of the BCS eigenenergies and eigenvalues of h_B . Physically, a BCS pair η_0 now becomes a dressed BCS pair $W\eta_0W^\dagger$ and observables have expectation values with the same form as (14). *All known BCS-type models can be generated by their own particular dressing transformations*. For instance, a dressing transformation

$$W = \exp(i \sum_k \phi_k c_{-k\downarrow}^\dagger c_{-k\downarrow}), \quad (19)$$

can put the BCS pairs into bath, i. e., a phonon bath with $h_B = \sum \omega_t b_t^\dagger b_t$, where ϕ_k are bath operators. When $\phi_k = \sum_k \lambda_k^t b_t^\dagger b_t$, the dressed Hamiltonian is

$$\begin{aligned} H &= \sum_{k,\sigma=\downarrow,\uparrow} \epsilon_k n_{k\sigma} + \sum \omega_t b_t^\dagger b_t \\ &+ G \sum_{k,k'} e^{i(\phi_{k'} - \phi_k)} \eta_0^\dagger(k') \eta_0(k). \end{aligned}$$

For weak coupling $\lambda_k^t \ll 1$,

$$H \approx H_0 + h_B + iG \sum_k \lambda_k^t b_t^\dagger b_t \eta_0^\dagger(k') \eta_0 + h.c., \quad (20)$$

the last term denotes a standard dephasing from the phonon bath. It shows that the BCS dynamics is naturally fault-tolerant against the dephasing.

Conclusion.— Using the dressing transformations, I have found an exactly solvable family of pairing models. The BCS pairs are peculiar in the family as they live in a three-dimensional spin space, while all other pairs survive on two dimensional spin space. The fact that high T_c superconductivity occurs on two dimensions ensures the legitimacy of the $q \neq 0$ components being a physically valid Hamiltonian. I anticipate that two dimensionality of these models and high T_c superconductivity is not coincident but what goes in nature. These pairing models are on equal footing but distinguishing themselves according to extra single particle energies. This seems to suggest that a d -wave pair is a dressed s -wave pair or vice versa, which might be a solution to the d -wave [1] or s -wave [2] theoretical issue. I look into an example and notice that the extra energies are responsible for the transitions between independent-particle states with zero magnetization and collective η pairing or d -wave pairing states. I also secure a condition where another type of pairs can play the role that the BCS pairs are playing. In addition, I emphasize that the

family of solvable models can be even much bigger.

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